

## G07BEF – NAG Fortran Library Routine Document

**Note.** Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

### 1 Purpose

G07BEF computes maximum likelihood estimates for parameters of the Weibull distribution from data which may be right-censored.

### 2 Specification

```

SUBROUTINE G07BEF(CENS, N, X, IC, BETA, GAMMA, TOL, MAXIT, SEBETA,
1              SEGAM, CORR, DEV, NIT, WK, IFAIL)
  INTEGER      N, IC(*), MAXIT, NIT, IFAIL
  real         X(N), BETA, GAMMA, TOL, SEBETA, SEGAM, CORR,
1              DEV, WK(N)
  CHARACTER*1  CENS

```

### 3 Description

G07BEF computes maximum likelihood estimates of the parameters of the Weibull distribution from exact or right-censored data.

For  $n$  realizations,  $y_i$ , from a Weibull distribution a value  $x_i$  is observed such that

$$x_i \leq y_i.$$

There are two situations:

- (a) exactly specified observations, when  $x_i = y_i$
- (b) right-censored observations, known by a lower bound, when  $x_i < y_i$ .

The probability density function of the Weibull distribution, and hence the contribution of an exactly specified observation to the likelihood, is given by:

$$f(x; \lambda, \gamma) = \lambda \gamma x^{\gamma-1} \exp(-\lambda x^\gamma), \quad x > 0, \quad \text{for } \lambda, \gamma > 0;$$

while the survival function of the Weibull distribution, and hence the contribution of a right-censored observation to the likelihood, is given by:

$$S(x; \lambda, \gamma) = \exp(-\lambda x^\gamma), \quad x > 0, \quad \text{for } \lambda, \gamma > 0.$$

If  $d$  of the  $n$  observations are exactly specified and indicated by  $i \in D$  and the remaining  $(n - d)$  are right-censored, then the likelihood function,  $Like(\lambda, \gamma)$  is given by

$$Like(\lambda, \gamma) \propto (\lambda \gamma)^d \left( \prod_{i \in D} x_i^{\gamma-1} \right) \exp \left( -\lambda \sum_{i=1}^n x_i^\gamma \right).$$

To avoid possible numerical instability a different parameterization  $\beta, \gamma$  is used, with  $\beta = \log(\lambda)$ . The kernel log-likelihood function,  $L(\beta, \gamma)$ , is then:

$$L(\beta, \gamma) = d \log(\gamma) + d\beta + (\gamma - 1) \sum_{i \in D} \log(x_i) - e^\beta \sum_{i=1}^n x_i^\gamma.$$

If the derivatives  $\frac{\partial L}{\partial \beta}$ ,  $\frac{\partial L}{\partial \gamma}$ ,  $\frac{\partial^2 L}{\partial \beta^2}$ ,  $\frac{\partial^2 L}{\partial \beta \partial \gamma}$  and  $\frac{\partial^2 L}{\partial \gamma^2}$  are denoted by  $L_1$ ,  $L_2$ ,  $L_{11}$ ,  $L_{12}$  and  $L_{22}$ , respectively, then the maximum likelihood estimates,  $\hat{\beta}$  and  $\hat{\gamma}$ , are the solution to the equations:

$$L_1(\hat{\beta}, \hat{\gamma}) = 0 \tag{1}$$

and

$$L_2(\hat{\beta}, \hat{\gamma}) = 0 \quad (2)$$

Estimates of the asymptotic standard errors of  $\hat{\beta}$  and  $\hat{\gamma}$  are given by:

$$\text{se}(\hat{\beta}) = \sqrt{\frac{-L_{22}}{L_{11}L_{22} - L_{12}^2}}, \quad \text{se}(\hat{\gamma}) = \sqrt{\frac{-L_{11}}{L_{11}L_{22} - L_{12}^2}}.$$

An estimate of the correlation coefficient of  $\hat{\beta}$  and  $\hat{\gamma}$  is given by:

$$\frac{L_{12}}{\sqrt{L_{11}L_{22}}}.$$

**Note.** If an estimate of the original parameter  $\lambda$  is required, then

$$\hat{\lambda} = \exp(\hat{\beta}) \quad \text{and} \quad \text{se}(\hat{\lambda}) = \hat{\lambda} \text{se}(\hat{\beta}).$$

The equations (1) and (2) are solved by the Newton–Raphson iterative method with adjustments made to ensure that  $\hat{\gamma} > 0.0$ .

## 4 References

- [1] Gross A J and Clark V A (1975) *Survival Distributions: Reliability Applications in the Biomedical Sciences* Wiley
- [2] Kalbfleisch J D and Prentice R L (1980) *The Statistical Analysis of Failure Time Data* Wiley

## 5 Parameters

- 1: CENS — CHARACTER\*1 *Input*  
*On entry:* indicates whether the data is censored or non-censored.  
 If CENS = 'N', then each observation is assumed to be exactly specified. IC is not referenced.  
 If CENS = 'C', then each observation is censored according to the value contained in IC(*i*), for  $i = 1, 2, \dots, n$ .  
*Constraint:* CENS = 'C' or 'N'.
- 2: N — INTEGER *Input*  
*On entry:* the number of observations,  $n$ .  
*Constraint:*  $N \geq 1$ .
- 3: X(N) — *real* array *Input*  
*On entry:* X(*i*) contains the *i*th observation,  $x_i$ , for  $i = 1, 2, \dots, n$ .  
*Constraint:*  $X(i) > 0.0$ , for  $i = 1, 2, \dots, n$ .
- 4: IC(\*) — INTEGER array *Input*  
**Note.** If CENS = 'C', then IC must be dimensioned at least N, otherwise IC can be dimensioned 1.  
*On entry:* if CENS = 'C', then IC(*i*) contains the censoring codes for the *i*th observation, for  $i = 1, 2, \dots, n$ .  
 If IC(*i*) = 0, the *i*th observation is exactly specified.  
 If IC(*i*) = 1, the *i*th observation is right-censored.  
 If CENS = 'N', then IC is not referenced.  
*Constraint:* if CENS = 'C', then IC(*i*) = 0 or 1, for  $i = 1, 2, \dots, n$ .

- 5:** BETA — *real* *Output*  
*On exit:* the maximum likelihood estimate,  $\hat{\beta}$ , of  $\beta$ .
- 6:** GAMMA — *real* *Input/Output*  
*On entry:* indicates whether an initial estimate of  $\gamma$  is provided.  
 If GAMMA > 0.0, it is taken as the initial estimate of  $\gamma$  and an initial estimate of  $\beta$  is calculated from this value of  $\gamma$ .  
 If GAMMA  $\leq$  0.0, then initial estimates of  $\gamma$  and  $\beta$  are calculated, internally, providing the data contains at least two distinct exact observations. (If there are only two distinct exact observations, then the largest observation must not be exactly specified.) See Section 8 for further details.  
*On exit:* contains the maximum likelihood estimate,  $\hat{\gamma}$ , of  $\gamma$ .
- 7:** TOL — *real* *Input*  
*On entry:* the relative precision required for the final estimates of  $\beta$  and  $\gamma$ . Convergence is assumed when the absolute relative changes in the estimates of both  $\beta$  and  $\gamma$  are less than TOL.  
 If TOL = 0.0, then a relative precision of 0.000005 is used.  
*Constraint:* *machine precision*  $\leq$  TOL  $\leq$  1.0 or TOL = 0.0.
- 8:** MAXIT — INTEGER *Input*  
*On entry:* the maximum number of iterations allowed.  
 If MAXIT  $\leq$  0, then a value of 25 is used.
- 9:** SEBETA — *real* *Output*  
*On exit:* an estimate of the standard error of  $\hat{\beta}$ .
- 10:** SEGAM — *real* *Output*  
*On exit:* an estimate of the standard error of  $\hat{\gamma}$ .
- 11:** CORR — *real* *Output*  
*On exit:* an estimate of the correlation between  $\hat{\beta}$  and  $\hat{\gamma}$ .
- 12:** DEV — *real* *Output*  
*On exit:* the maximized kernel log-likelihood,  $L(\hat{\beta}, \hat{\gamma})$ .
- 13:** NIT — INTEGER *Output*  
*On exit:* the number of iterations performed.
- 14:** WK(N) — *real* array *Workspace*
- 15:** IFAIL — INTEGER *Input/Output*  
*On entry:* IFAIL must be set to 0, -1 or 1. For users not familiar with this parameter (described in Chapter P01) the recommended value is 0.  
*On exit:* IFAIL = 0 unless the routine detects an error (see Section 6).

## 6 Error Indicators and Warnings

If on entry `IFAIL = 0` or `-1`, explanatory error messages are output on the current error message unit (as defined by `X04AAF`).

Errors detected by the routine:

`IFAIL = 1`

- On entry, `CENS`  $\neq$  'N' or 'C',
- or `N`  $< 1$ ,
- or `TOL`  $< 0.0$ ,
- or  $0.0 < \text{`TOL`} < \textit{machine precision}$ ,
- or `TOL`  $> 1.0$ .

`IFAIL = 2`

- On entry, the  $i$ th observation,  $X(i) \leq 0.0$ , for some  $i = 1, 2, \dots, n$ ,
- or the  $i$ th censoring code,  $IC(i) \neq 0$  or  $1$ , for some  $i = 1, 2, \dots, n$  and `CENS` = 'C'.

`IFAIL = 3`

On entry, there are no exactly specified observations, or the routine was requested to calculate initial values and there are either less than two distinct exactly specified observations or there are exactly two and the largest observation is one of the exact observations.

`IFAIL = 4`

The method has failed to converge in `MAXIT` iterations. The user should increase `TOL` or `MAXIT`.

`IFAIL = 5`

Process has diverged. The process is deemed divergent if three successive increments of  $\beta$  or  $\gamma$  increase or if the Hessian matrix of the Newton–Raphson process is singular. Either different initial estimates should be provided or the data should be checked to see if the Weibull distribution is appropriate.

`IFAIL = 6`

A potential overflow has been detected. This is an unlikely exit usually caused by a large input estimate of  $\gamma$ .

## 7 Accuracy

Given that the Weibull distribution is a suitable model for the data and that the initial values are reasonable the convergence to the required accuracy, indicated by `TOL`, should be achieved.

## 8 Further Comments

The initial estimate of  $\gamma$  is found by calculating a Kaplan–Meier estimate of the survival function,  $\hat{S}(x)$ , and estimating the gradient of the plot of  $\log(-\log(\hat{S}(x)))$  against  $x$ . This requires the Kaplan–Meier estimate to have at least two distinct points.

The initial estimate of  $\hat{\beta}$ , given a value of  $\hat{\gamma}$ , is calculated as

$$\hat{\beta} = \log \left( \frac{d}{\sum_{i=1}^n x_i^{\hat{\gamma}}} \right).$$

## 9 Example

In a study, 20 patients receiving an analgesic to relieve headache pain had the following recorded relief times (in hours): 1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7 4.1 1.8 1.5 1.2 1.4 3.0 1.7 2.3 1.6 2.0 (See Gross and Clarke [1]). This data is read in and a Weibull distribution fitted assuming no censoring; the parameter estimates and their standard errors are printed.

## 9.1 Program Text

**Note.** The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```

*      G07BEF Example Program Text
*      Mark 15 Release. MAG Copyright 1991.
*      .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER       (NIN=5,NOUT=6)
      INTEGER          NMAX
      PARAMETER       (NMAX=20)
*      .. Local Scalars ..
      real            BETA, CORR, DEV, GAMMA, SEBETA, SEGAM, TOL
      INTEGER          I, IFAIL, MAXIT, N, NIT
*      .. Local Arrays ..
      real            WK(NMAX), X(NMAX)
      INTEGER          IC(NMAX)
*      .. External Subroutines ..
      EXTERNAL         G07BEF
*      .. Executable Statements ..
      WRITE (NOUT,*) 'G07BEF Example Program Results'
*      Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N
      IF (N.LE.NMAX) THEN
          READ (NIN,*) (X(I),I=1,N)
*          If data were censored then IC would also be read in.
*          Leave G07BEF to calculate initial values
          GAMMA = 0.0e0
*          Use default values for TOL and MAXIT
          TOL = 0.0e0
          MAXIT = 0
          IFAIL = 0
*
          CALL G07BEF('No censor',N,X,IC,BETA,GAMMA,TOL,MAXIT,SEBETA,
+                  SEGAM,CORR,DEV,NIT,WK,IFAIL)
*
          WRITE (NOUT,*)
          WRITE (NOUT,99999) ' BETA = ', BETA, ' Standard error = ',
+              SEBETA
          WRITE (NOUT,99999) ' GAMMA = ', GAMMA, ' Standard error = ',
+              SEGAM
          END IF
      STOP
*
      99999 FORMAT (1X,2(A,F10.4))
      END

```

## 9.2 Program Data

```

G07BEF Example Program Data
20
1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7
4.1 1.8 1.5 1.2 1.4 3.0 1.7 2.3 1.6 2.0

```

### 9.3 Program Results

G07BEF Example Program Results

BETA	=	-2.1073	Standard error =	0.4627
GAMMA	=	2.7870	Standard error =	0.4273

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